J. Fluid Mech. (1984), *vol.* 145, *pp.* 405–415
Printed in Great Britain

Small-amplitude free-surface waves generated by moving oscillatory disturbances

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(Received 3 October 1983 and in revised form 27 February 1984)

The problem of capillary–gravity waves generated by certain moving oscillatory surface-pressure distributions is investigated. The main difficulty of the problem lies in finding the real roots of the modified frequency equations. This is dealt with by the use of certain geometric considerations. The critical condition that results from the formation of double roots of the modified frequency equations is represented as a surface. This surface divides the whole space into several distinct regions. For points in different regions the propagation of waves is different. The waves are determined in all cases.

1. Introduction

The two-dimensional problem of water waves generated by a moving oscillatory pressure distribution has been investigated in many papers. Thus the problem without surface tension has been studied by Kaplan (1957) in a fluid of infinite depth, and by Debnath & Rosenblat (1969) in a fluid with finite uniform depth. The problem with infinite depth has been reinvestigated in greater detail by Magnuson (1977). The problem with surface tension has been investigated by Pramanik (1980) in a fluid of infinite depth. The aim of the present paper is to study the problem with surface tension in a fluid with finite uniform depth.

The main difficulty of the problem lies in finding out the real roots of certain characteristic equations called the modified frequency equations, and upon these roots the resulting wave pattern crucially depends. Pramanik (1980) has shown how with the help of geometrical considerations the complete set of roots of the frequency equations can be determined and their effects on the resulting waves can be ascertained.

When the depth of the fluid is infinite there exist two dimensionless parameters that characterize the wave pattern. Supposing ω to be the frequency of oscillation, V the uniform velocity of the pressure distribution, T' the surface tension and ρ the density of the fluid, the two parameters are $\alpha = V\omega/g$ and $\beta = V/(Tg)^{\frac{1}{4}}$, where $T = T'/\rho$. The critical case that occurs when two roots of the frequency equations coalesce can be represented by a curve in the (α, β)-plane; this curve divides the positive quadrant of the (α, β)-plane into several distinct regions such that for points (α, β) in various regions the propagation of waves is different.

Now when the depth of the fluid is finite, there are three dimensionless parameters $a = V\omega/g$, $b = V/(gh)^{\frac{1}{2}}$ and $c = T/h^2g$, where h is the depth, and the critical condition is a surface $f(a, b, c) = 0$. It is easy to understand by analogy with the infinite-depth case that this critical surface will divide the positive quadrant of the (a, b, c)-space into several distinct regions, and for points (a, b, c) in different regions the propagation of waves is different. But the complete determination of these regions is rather a difficult task. However, it is found that the intersecting curves of the critical surface by the plane $c =$ constant have only two forms. For $0 < c < \frac{1}{3}$ this intersecting curve divides the plane of the curve into five distinct regions, in each of which the set of roots of the frequency equations is different, while for any $c \geqslant \frac{1}{3}$ the corresponding curve divides the plane into two such regions. In the former case the wave pattern is similar to the infinite-depth case. In the ultimate steady state there exist six waves: four gravity waves and two capillary waves. Among them, three gravity waves propagate in the downstream direction and one gravity wave exists in the upstream side, while both the capillary waves exist in the upstream side. In the latter case the total number of waves is four: two gravity waves in the downstream side and two capillary waves in the upstream side. The solution becomes singular for (a, b, c) on the critical surfaces.

The case $\omega = 0$ is considered. The corresponding problem of gravity waves was considered by Stoker (1957), who found that the steady state consists of a wave in the downstream direction and that the critical case is $b = 1$. It is found here that the surface tension, as well as modifying the critical case, introduces one wave in the upstream side.

We have formulated the problem as a linearized initial-value problem and have solved it by Fourier-transform and asymptotic methods. Some of the results could be obtained through a kinematic approach using the concept of group velocity as described by Whitham (1974). But, as remarked by Whitham, the full description of the wave pattern can be achieved only through applying the transform method to the full set of equations. Also, we prefer the initial-value formulation because by this we obtain the singular behaviour of the waves on the critical surface.

Though we have used the travelling oscillatory surface-pressure distribution as the wave-generating mechanism, it is clear that any travelling oscillatory disturbance would produce a similar wave pattern and could be determined following our approach.

2. Formulation

We take a moving coordinate system Oxy such that Oy is vertically upwards, and with the origin located on the undisturbed free surface and moving with uniform velocity V along the positive direction of the x-axis. The system being initially at rest, the waves are generated by the continued application of the pressure distribution $f(x)\exp(\mathrm{i}\omega t)$, which at the same time moves with velocity V. Let $\phi(x, y, t)$ be the velocity potential and $\eta(x, t)$ the surface elevation. Then we have the following initial-value problem:

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{for} \quad -\infty < x < \infty, \quad -h \leqslant y \leqslant 0, \quad t \geqslant 0, \tag{1}$$

$$\left.\begin{aligned} \eta_t - V\eta_x &= \phi_y, && (2)\\ \phi_t - V\phi_x + g\eta + \frac{1}{\rho} f(x)\,\mathrm{e}^{\mathrm{i}\omega t} &= T\eta_{xx} && (3) \end{aligned}\right\} \quad \text{at} \quad y = 0,$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y = -h, \tag{4}$$

$$\phi(x, y, 0) = 0, \quad \eta(x, 0) = 0. \tag{5}$$

The formal solution of the problem is obtained by Fourier-transforming the above system of equations with respect to x and then using the Fourier inversion formula. The following integral representation for η can easily be obtained:

$$\eta = \int_0^\infty \frac{\kappa \bar{f}(\kappa)}{\sigma(\sigma-\omega+\kappa V)} e^{i(\sigma+\kappa V)t+i\kappa x} d\kappa + \int_0^\infty \frac{\kappa \bar{f}(\kappa)}{\sigma(\sigma+\omega-\kappa V)} e^{i(-\sigma+\kappa V)t+i\kappa x} d\kappa$$

$$+\int_0^\infty \frac{\kappa \bar{f}(-\kappa)}{\sigma(\sigma-\omega-\kappa V)} e^{i(\sigma-\kappa V)t-i\kappa x} d\kappa + \int_0^\infty \frac{\kappa \bar{f}(-\kappa)}{\sigma(\sigma+\omega+\kappa V)} e^{-i(\sigma+\kappa V)t-i\kappa x} d\kappa$$

$$-e^{i\omega t}\int_0^\infty \frac{\kappa \bar{f}(\kappa)}{\sigma(\sigma-\omega+\kappa V)} e^{i\kappa x} d\kappa - e^{i\omega t}\int_0^\infty \frac{\kappa \bar{f}(\kappa)}{\sigma(\sigma+\omega-\kappa V)} e^{i\kappa x} d\kappa$$

$$-e^{i\omega t}\int_0^\infty \frac{\kappa \bar{f}(-\kappa)}{\sigma(\sigma-\omega-\kappa V)} e^{-i\kappa x} d\kappa - e^{i\omega t}\int_0^\infty \frac{\kappa \bar{f}(-\kappa)}{\sigma(\sigma+\omega+\kappa V)} e^{-i\kappa x} d\kappa, \tag{6a}$$

where

$$\sigma = \{(g\kappa + T\kappa^3)\tanh \kappa h\}^{\frac{1}{2}}, \quad \bar{f}(\kappa) = \frac{1}{4\pi\rho}\int_{-\infty}^{\infty} f(x)\, e^{-i\kappa x}\, dx. \tag{6b}$$

3. Steady-state waves

We shall determine the asymptotic values of the integrals for large times at a large distance from the pressure segment. These asymptotic values appear as contributions in the form of residues at the real poles of the integrands in (6). These poles are the solutions of the following three equations:

$$\sigma - \kappa V - \omega = 0, \tag{7}$$

$$\sigma - \kappa V + \omega = 0, \tag{8}$$

$$\sigma + \kappa V - \omega = 0. \tag{9}$$

We shall determine these roots for given values of the parameters T, V, g and h. The roots will be determined as points of intersection of the curve $m = \sigma$ and the straight lines $m = \kappa V + \omega$, $m = \kappa V - \omega$ and $m = -\kappa V + \omega$. It is easy to see that for $0 < c < \frac{1}{3}$ the curve $m = \sigma$ has a point of inflexion at some point $\kappa = \kappa_0 > 0$, say, and that $d^2\sigma/d\kappa^2$ changes sign from negative to positive as κ increases through κ_0. So in this case the curve has the shape shown in figures 1 and 2. When, however, $c \geqslant \frac{1}{3}$ there is no point of inflexion, and $d^2\sigma/d\kappa^2 > 0$ for all κ. So the curve here is as shown in figure 5. Since the roots of (7)–(9) and hence the resulting wave pattern depend crucially upon the shape of the curve, we divide our subsequent discussion into two parts, for $0 < c < \frac{1}{3}$ and for $c \geqslant \frac{1}{3}$.

3.1. *Steady waves for* $0 < c < \frac{1}{3}$

It can be seen from figure 1 that (7) can have a maximum of three distinct real positive roots, $\kappa = \alpha_1, \alpha_2, \alpha_3$, say, and from figure 2 that (8) can have two such roots, $\kappa = \alpha_4$, α_5. It is also easy to understand that (9) always has one positive root, $\kappa = \alpha_6$. To determine the exact set of roots for given values of the parameters, we consider the cases of repeated roots. It can be seen from figures 1 and 2 that there can be three

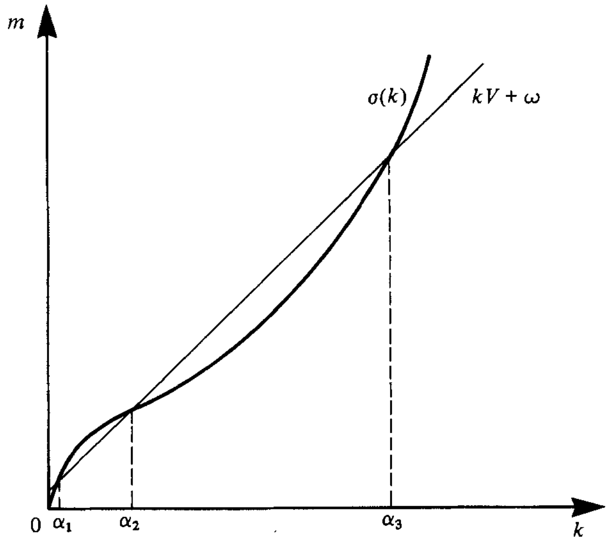

FIGURE 1. The roots of (7) for $c < \frac{1}{3}$.

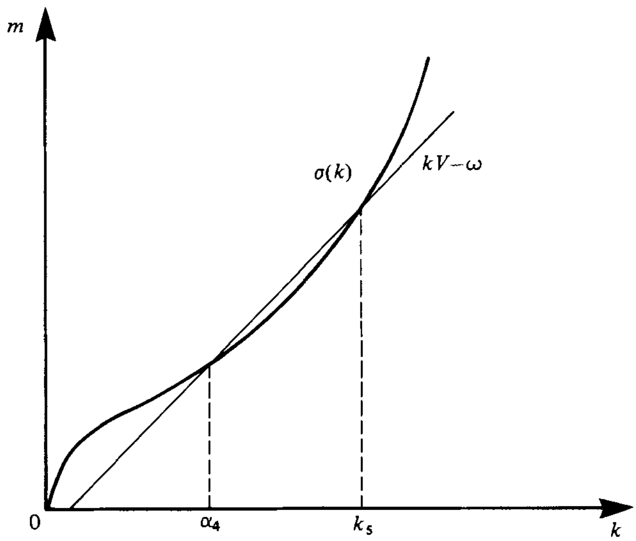

FIGURE 2. The roots of (8) for $c < \frac{1}{3}$.

such cases: $\alpha_1 = \alpha_2$, $\alpha_2 = \alpha_3$, and $\alpha_4 = \alpha_5$. The precise conditions for the occurrence of these cases can be written respectively as

$$\left.\begin{aligned} \sigma &= \kappa V + \omega, \\ \sigma' &= V \end{aligned}\right\} \quad 0 \leqslant \kappa < \kappa_0, \tag{10}$$

$$\left.\begin{aligned} \sigma &= \kappa V + \omega, \\ \sigma' &= V \end{aligned}\right\} \quad \kappa_0 \leqslant \kappa < \kappa_1, \tag{11}$$

$$\left.\begin{aligned} \sigma &= \kappa V - \omega, \\ \sigma' &= V \end{aligned}\right\} \quad \kappa_1 \leqslant \kappa < \infty, \tag{12}$$

where σ' denotes the derivative of σ with respect to κ, and κ_1 is the value of κ for which the straight line $m = \kappa V$ is a tangent to $m = \sigma$. Now we write $\kappa h = \lambda$ and introduce the following three dimensionless parameters:

$$a = \frac{V\omega}{g}, \quad b = \frac{V}{(gh)^{\frac{1}{2}}}, \quad c = \frac{T}{h^2 g}. \tag{13}$$

In terms of these parameters, (10)–(12) can be put together in the following form:

$$a = \begin{cases} \dfrac{[(1-\lambda^2c)\tanh\lambda - \lambda(1+\lambda^2c)\,\mathrm{sech}^2\lambda][(1+3\lambda^2c)\tanh\lambda + \lambda(1+\lambda^2c)\,\mathrm{sech}^2\lambda]}{4(1+\lambda^2c)\tanh\lambda} \\ \qquad\qquad \text{for} \quad 0 < \lambda < \lambda_0 \quad \text{and} \quad \lambda_0 \leqslant \lambda < \lambda_1, \\ \dfrac{[\lambda(1+\lambda^2c)\,\mathrm{sech}^2\lambda - (1-\lambda^2c)\tanh\lambda][(1+3\lambda^2c)\tanh\lambda + \lambda(1+\lambda^2c)\,\mathrm{sech}^2\lambda]}{4(1+\lambda^2c)\tanh\lambda} \\ \qquad\qquad \text{for} \quad \lambda_1 \leqslant \lambda < \infty, \end{cases} \tag{14}$$

$$b = \frac{(1+3\lambda^2c)\tanh\lambda + \lambda(1+\lambda^2c)\,\mathrm{sech}^2\lambda}{2[\lambda(1+\lambda^2c)\tanh\lambda]^{\frac{1}{2}}} \quad \text{for} \quad 0 \leqslant \lambda < \infty. \tag{15}$$

In the above expressions λ_0 and λ_1 are the values of λ corresponding to $\kappa = \kappa_0$ and $\kappa = \kappa_1$, and they respectively satisfy the following equations:

$$3\lambda^2(1+\lambda^2c)\tanh^4\lambda - 2\lambda(1+\lambda^2c)(1+3\lambda^2c)\tanh^3\lambda + [12\lambda^2c(1+\lambda^2c) - 2\lambda^2(1+\lambda^2c)^2 - (1+3\lambda^2c)]\tanh^2\lambda + 2\lambda(1+\lambda^2c)(1+3\lambda^2c)\tanh\lambda - \lambda^2(1+\lambda^2c)^2 = 0, \tag{16}$$

$$(1-\lambda^2c)\tanh\lambda - \lambda(1+\lambda^2c)\,\mathrm{sech}^2\lambda = 0. \tag{17}$$

Now (14) and (15) represent in the parametric form a surface called the critical surface $f(a,b,c) = 0$ in the (a,b,c)-space. The intersecting curve of this surface by the plane c = constant can be drawn for certain values of c. For $c = 0.01$ this curve is shown in figure 3. The point A_0 corresponds to $\lambda = \lambda_0$ and the point A_1 to $\lambda = \lambda_1$. The curvilinear portions c_1, c_2 and c_3, extending respectively from the point A to A_0, from A_0 to A_1 and from A_1 to B (the point at infinity), represent the cases $\alpha_1 = \alpha_2$, $\alpha_2 = \alpha_3$ and $\alpha_4 = \alpha_5$ respectively. Now these critical curves and the straight line OA_0 divide the whole positive quadrant of the plane of the curve into seven distinct regions R_n $(n = 1, \ldots, 7)$ as shown in figure 3. In each of these regions there corresponds a definite subset of the roots α_n in the sense that the values of the parameters V, ω, g, T and h which determine these regions also fix the roots. The roots corresponding to each region can be determined in the following way.

For a fixed value of ω, as we increase V from zero, in the (a,b)-plane the point (a,b) moves along a straight line through the origin and with constant gradient $\omega h^{\frac{1}{2}}/g^{\frac{1}{2}}$, while in the (m,κ)-plane the straight line $m = \kappa V + \omega$ rotates about a fixed point on the m-axis. The straight line OA_0 has the gradient $\omega_0 h^{\frac{1}{2}}/g^{\frac{1}{2}}$, where $\sigma(\kappa_0) = \omega_0$. For a fixed value of $\omega < \omega_0$, as V increases, the point (a,b) intersects all the curves c_1, c_2 and c_3, while for $\omega > \omega_0$ only c_3 is intersected. The corresponding situation in the (m,κ)-plane is easily understood. For $\omega > \omega_0$ the straight line $m = \kappa V + \omega$ is never tangent to the curve $m = \sigma$, while for any fixed $\omega < \omega_0$ it becomes tangent to $m = \sigma$ for two values of V. Similarly, for each value of ω the straight line $m = \kappa V - \omega$ is tangent to $m = \sigma$ for one value of V. We now turn our attention to the roots of (7). For a fixed $\omega < \omega_0$, as we increase V from zero, in the (a,b)-plane we first meet c_2

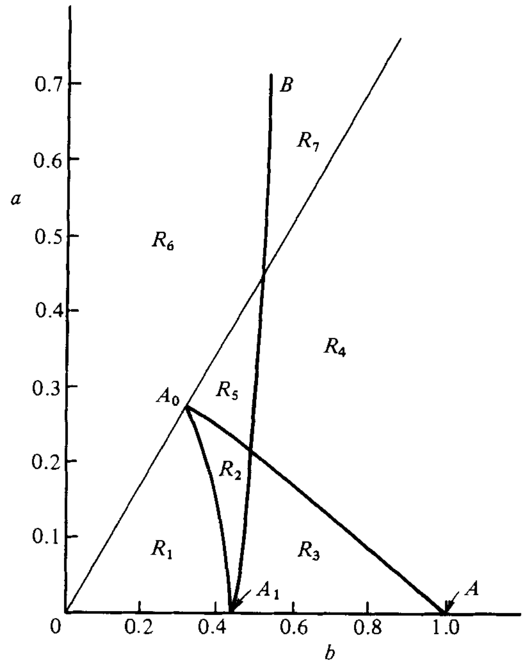

FIGURE 3. The section of the critical surface for $c = 0.01$.

and then c_1. That is, in the (m, κ)-plane as the straight line $m = \kappa V + \omega$ is rotated about a point on the m-axis below ω_0 from a position in which it is parallel to the κ-axis, the straight line at first remains in a position in which it intersects $m = \sigma$ at one point $\kappa = \alpha_1$, then it comes to a position in which it is tangent to the curve such that $\alpha_3 = \alpha_2$ occurs, and thereafter it is in a position in which all three roots α_1, α_2 and α_3 exist up to the position at which it again becomes tangent such that $\alpha_1 = \alpha_2$ occurs, and then only $\kappa = \alpha_3$ exists. Thus for points (a, b) in the regions to the left of c_2 only α_1 exists, in the regions to the left of c_1 but to the right of c_2 all three roots α_1, α_2 and α_3 exist, and in the regions to the right of c_1 only α_3 exists. In a similar way it can easily be verified that the roots α_4 and α_5 of (8) exist only for points (a, b) to the right of c_3. Thus we arrive at the following distributions of roots:

R_1	R_2	R_3	R_4	R_5	R_6	R_7
α_1, α_6	α_1, α_2	$\alpha_1, \alpha_2, \alpha_3$	α_3, α_4	α_3, α_6	α_3, α_6	α_3, α_4
	α_3, α_6	$\alpha_4, \alpha_5, \alpha_6$	α_5, α_6			α_5, α_6

We note that the roots in the regions R_5 and R_6, and also in R_4 and R_7, are identical. Thus the straight line OA_0 is effective up to the point A_0, and there are only five distinct regions.

Once the poles are located, the asymptotic values of the integrals can be determined by a method developed by Lighthill (1958). The behaviour of the asymptotic waves for (a, b) in any of the regions where the poles are distinct, as well as for (a, b) on the critical curves where some poles are double, can be determined by this method. As an illustration we consider the integral

$$I = \int_0^\infty \frac{\kappa \bar{f}(-\kappa)\, \mathrm{e}^{-\mathrm{i}\kappa x}}{\sigma(\sigma - \omega - \kappa V)} \mathrm{e}^{\mathrm{i}(\sigma - \kappa V)t}\, \mathrm{d}\kappa. \tag{18}$$

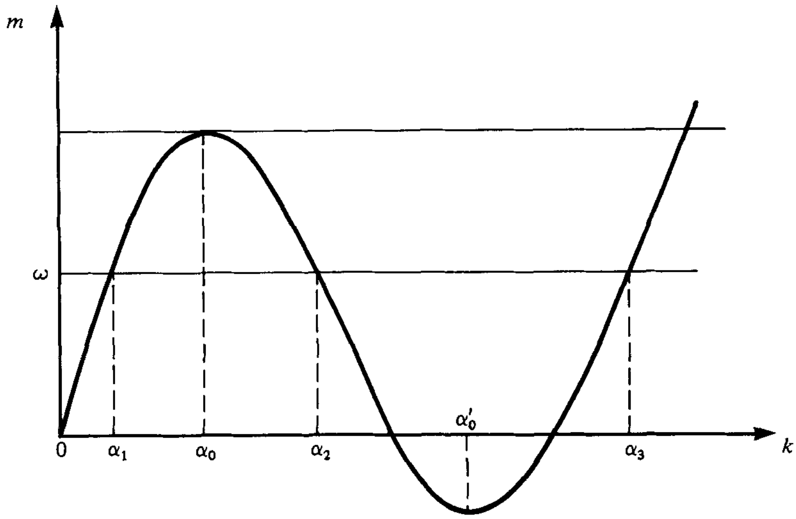

FIGURE 4. The form of the curve defined in (19).

This is to be evaluated for $t \to \infty$. Let us at first suppose that (a, b) is within the region R_3, so that the poles α_1, α_2 and α_3 are all distinct. A suitable transformation for this case is

$$m = m(\kappa) = \sigma - \kappa V. \tag{19}$$

A sketch of the curve of the transformation is shown in figure 4. The poles α_1, α_2 and α_3 are the points of intersection of the curve $m = \sigma - \kappa V$ and the straight line $m = \omega$. The points $\kappa = \alpha_0$ and $\kappa = \alpha_0'$ are the solutions of the equation $\sigma' = V$. The curve $m = \sigma - \kappa V$ may have other forms. But it can be proved that in the case under consideration α_0 and α_0' always exist and $m(\alpha_0) > m(\alpha_0')$. Now to calculate the asymptotic value of I as $t \to \infty$ we break it up as follows:

$$\begin{aligned} I &= \left(\int_0^{\alpha_0} + \int_{\alpha_0}^{\alpha_0'} + \int_{\alpha_0'}^{\infty}\right)\left[\frac{\kappa \bar{f}(-\kappa)\,\mathrm{e}^{-\mathrm{i}\kappa x}}{\sigma(\sigma - \kappa V - \omega)}\right]\mathrm{e}^{\mathrm{i}(\sigma - \kappa V)t}\,\mathrm{d}\kappa \\ &= \left(\int_0^{m(\alpha_0)} - \int_{m(\alpha_0')}^{m(\alpha_0)} + \int_{m(\alpha_0')}^{\infty}\right)\left[\frac{\kappa \bar{f}(-\kappa)\,\mathrm{e}^{-\mathrm{i}\kappa x}}{\sigma}\frac{\mathrm{d}\kappa}{\mathrm{d}m}\right]\frac{\mathrm{e}^{\mathrm{i}mt}}{m - \omega}\,\mathrm{d}m, \end{aligned} \tag{20}$$

where the quantity within the square bracket is to be expressed in terms of m by (19). Now each of the integrals in (20) contains one pole. Hence the formula referred to above gives the following asymptotic value for $t \to \infty$:

$$\begin{aligned} I \simeq{}& \frac{\pi \mathrm{i} \alpha_1 \bar{f}(-\alpha_1)}{\sigma(\alpha_1)\left[\sigma'(\alpha_1) - V\right]}\mathrm{e}^{\mathrm{i}(\omega t - \alpha_1 x)} - \frac{\pi \mathrm{i} \alpha_2 \bar{f}(-\alpha_2)}{\sigma(\alpha_2)\left[\sigma'(\alpha_2) - V\right]}\mathrm{e}^{\mathrm{i}(\omega t - \alpha_2 x)} \\ &+ \frac{\pi \mathrm{i} \alpha_3 \bar{f}(-\alpha_3)}{\sigma(\alpha_3)\left[\sigma'(\alpha_3) - V\right]}\mathrm{e}^{\mathrm{i}(\omega t - \alpha_3 x)}. \end{aligned}$$

When (a, b) is either on the curve c_1 or on the curve c_2, two of the roots coincide. Let us calculate the integral for (a, b) on c_1. Here $\alpha_1 = \alpha_2 = \alpha_0$ and α_3 is distinct. We break up the integral as follows:

$$I = \left(\int_0^{\alpha_0 - \epsilon} + \int_{\alpha_0 - \epsilon}^{\alpha_0} + \int_{\alpha_0}^{\alpha_0 + \epsilon} + \int_{\alpha_0 + \epsilon}^{\alpha_0'} + \int_{\alpha_0'}^{\infty}\right)\left[\frac{\kappa \bar{f}(-\kappa)\,\mathrm{e}^{-\mathrm{i}\kappa x}}{\sigma(\sigma - \kappa V - \omega)}\right]\mathrm{e}^{\mathrm{i}(\sigma - \kappa V)t}\,\mathrm{d}\kappa, \tag{21}$$

where $0 < \epsilon \ll 1$.

The first and the fourth integrals, having no poles, will not contribute to the asymptotic value, and the contribution of the last integral from the distinct pole $\kappa = \alpha_3$ can be calculated as above. To calculate the second and the third integrals we make the substitution

$$m = \kappa V - \sigma + \omega. \tag{22}$$

Now near $\kappa = \alpha_0$ we can write

$$m = -\tfrac{1}{2}(\kappa - \alpha_0)^2 \sigma''(\alpha_0).$$

Also we get

$$\frac{\mathrm{d}m}{\mathrm{d}\kappa} = |2m\sigma''(\alpha_0)|^{\frac{1}{2}} \operatorname{sgn}(\kappa - \alpha_0).$$

Substituting all these, the combination of the second and the third integrals in (21) reduces to

$$-2\mathrm{e}^{\mathrm{i}\omega t} \int_0^{\epsilon_1} |2\sigma''(\alpha_0)|^{-\frac{1}{2}} \left[\frac{\kappa \bar{f}(-\kappa)\,\mathrm{e}^{-\mathrm{i}\kappa x}}{\sigma}\right] \frac{\mathrm{e}^{-\mathrm{i}mt}}{|m|^{\frac{3}{2}}}\, \mathrm{d}m, \tag{23}$$

where ϵ_1 is a positive quantity.

The asymptotic value of the integral in (23) remains unchanged if the upper limit is replaced by ∞, and hence by Lighthill's formula the asymptotic value is given by

$$-2\Gamma(-\tfrac{1}{2})\,|2\sigma''(\alpha_0)|^{-\frac{1}{2}} \frac{\alpha_0 \bar{f}(-\alpha_0)}{\sigma(\alpha_0)}\, t^{\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}(\omega t - \alpha_0 x) + \frac{1}{4}\mathrm{i}\pi} \quad \text{as} \quad t \to \infty. \tag{24}$$

It is thus seen that the asymptotic value of I becomes infinitely large for (a, b) on c_1. In a similar way, it can be proved that for (a, b) on any of the critical curves some of the integrals in $\eta(x, t)$ have infinitely large values. Thus the solution for $\eta(x, t)$ becomes singular for (a, b) on the critical curves.

We now turn our attention to the case when (a, b) is outside the critical curves, where the asymptotic solution is a system of steady waves. For (a, b) within R_3, the system of waves is as follows:

$$\eta = \begin{cases} \bar{f}(-\alpha_1)\, F(\alpha_1)\, \mathrm{e}^{\mathrm{i}(\omega t - \alpha_1 x)} + \bar{f}(-\alpha_3)\, F(\alpha_3)\, \mathrm{e}^{\mathrm{i}(\omega t - \alpha_3 x)} \\ \quad -\bar{f}(\alpha_5)\, F(\alpha_5)\, \mathrm{e}^{\mathrm{i}(\omega t + \alpha_5 x)} \quad \text{as} \quad x \to \infty, \\ \quad -\bar{f}(-\alpha_2)\, F(\alpha_2)\, \mathrm{e}^{\mathrm{i}(\omega t - \alpha_2 x)} + \bar{f}(\alpha_4)\, F(\alpha_4)\, \mathrm{e}^{\mathrm{i}(\omega t + \alpha_4 x)} \\ \quad +\bar{f}(\alpha_6)\, F_1(\alpha_6)\, \mathrm{e}^{\mathrm{i}(\omega t + \alpha_6 x)} \quad \text{as} \quad x \to -\infty, \end{cases} \tag{25}$$

where

$$F(\kappa) = \frac{2\pi \mathrm{i}\kappa}{\sigma(\sigma' - V)}, \quad F_1(\kappa) = \frac{2\pi \mathrm{i}\kappa}{\sigma(\sigma' + V)}.$$

It is convenient to refer to a wave that comes as a contribution from the pole α_n as an α_n wave. It is thus seen that when the point (a, b) is in R_3 there exist six waves. Among these the α_1, α_2, α_4 and α_6 waves are the gravity waves and the rest are the capillary waves. Three gravity waves exist in the downstream and one in the upstream side, while both the capillary waves exist in the upstream side. It is shown by Whitham (1974), using the concept of group velocity, that in a moving medium the capillary waves always exist ahead of the disturbance, which agrees with our findings. The waves for the points (a, b) in other regions are easy to determine. They are simply the waves expressed in (25), only with the wave coming from a pole not occurring in a region being deleted from that region.

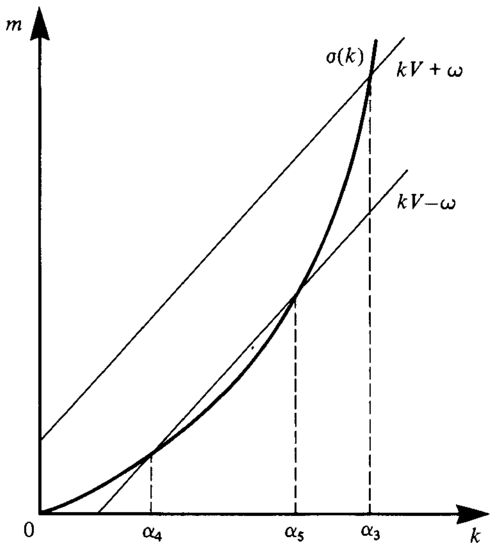

FIGURE 5. The roots of (7) and (8) for $c \geqslant \frac{1}{3}$.

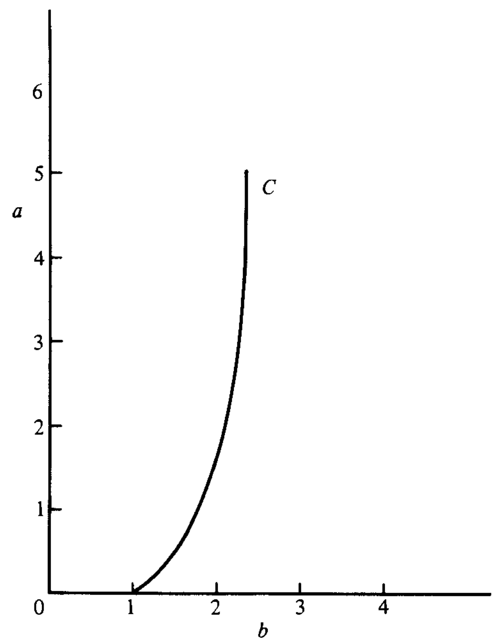

FIGURE 6. The section of the critical surface for $c = 1$.

3.2. *Steady waves for* $c \geqslant \frac{1}{3}$

It is evident from figure 5 that the roots of the frequency equations in the present case are α_3, α_4, α_5 and α_6. And α_4 and α_5 may coincide to give rise to the critical case which is represented as follows:

$$\left.\begin{aligned}\sigma' &= \kappa V - \omega,\\ \sigma' &= V\end{aligned}\right\} \quad \text{for} \quad 0 < \kappa < \infty.$$

This, with the help of transformation (13), is at once written down as

$$\left.\begin{aligned} a &= \frac{[\lambda(1+\lambda^2c)\operatorname{sech}^2\lambda-(1-\lambda^2c)\tanh\lambda][(1+3\lambda^2c)\tanh\lambda+\lambda(1+\lambda^2c)\operatorname{sech}^2\lambda],}{4(1+\lambda^2c)\tanh\lambda}, \\ b &= \frac{(1+3\lambda^2c)\tanh\lambda+\lambda(1+\lambda^2c)\operatorname{sech}^2\lambda}{2[\lambda(1+\lambda^2c)\tanh\lambda]^{\frac{1}{2}}} \end{aligned}\right\} \tag{26}$$

for $0<\lambda<\infty$. The intersecting curve of this critical surface is drawn for $c=1$ in figure 6. This curve C divides the corresponding plane into two regions. It is easy to verify that for points (a,b) to the right of the curve all the roots α_3, α_4, α_5 and α_6 exist, while for points (a,b) to the left of the curve only α_3 and α_6 exist. The steady-state value of η for points (a,b) to the right of the C is then given by

$$\eta=\left\{\begin{aligned} &\bar{f}(-\alpha_3)\,F(\alpha_3)\,e^{i(\omega t-\alpha_3 x)}-\bar{f}(\alpha_5)\,F(\alpha_5)\,e^{i(\omega t+\alpha_5 x)} \quad \text{as} \quad x\to\infty, \\ &\bar{f}(\alpha_4)\,F(\alpha_4)\,e^{i(\omega t+\alpha_4 x)}+\bar{f}(\alpha_6)\,F_1(\alpha_6)\,e^{i(\omega t+\alpha_6 x)} \quad \text{as} \quad x\to-\infty. \end{aligned}\right\} \tag{27}$$

For points (a,b) to the left of the curve only the α_3 and α_6 waves exist. Thus in this case, out of the four gravity waves only two exist, while both the capillary waves exist. The condition $c\geqslant\frac{1}{3}$ implies very shallow depth. For water this means that the depth is less than 0.5 cm. We see thus that in shallow water the capillary forces are more effective than the gravity forces.

4. Discussions

Though our discussions in §3 are based on two particular values of the parameter c, the results are valid in general. The qualitative nature of the solutions remains the same for any other values of c in the two ranges $(0,\frac{1}{3})$ and $(\frac{1}{3},\infty)$. For any other value of c in the range $(0,\frac{1}{3})$ only the points A_0 and A_1 on the critical curves will change positions. For values of c very near to zero, the point A_1 will be very near to the origin, and as c increases from zero the point A_1 moves along the b-axis towards the point A, and coincides with A as $c\to\frac{1}{3}$, so that the region bounded by the curves c_1, c_2 and the b-axis ultimately vanishes and we get the critical curve as in figure 6. All these statements are easily verified if we consider the section of the critical surface $f(a,b,c)=0$ by the plane $a=0$. The equation for this critical curve in the (b,c)-plane is obtained from the general equations (14) and (15). It can be seen from (14) that $a=0$ is satisfied when $\lambda=0$ for all values of c or when

$$(1-\lambda^2c)\tanh\lambda-\lambda(1+\lambda^2c)\operatorname{sech}^2\lambda=0.$$

This, with the help of (15), gives the following representation of the critical case for $a=0$;

$$b=1 \quad \text{for all } c,$$

and

$$\left.\begin{aligned} b &= \frac{\sqrt{2}\tanh\lambda}{[\lambda(\tanh\lambda+\lambda\operatorname{sech}^2\lambda)]^{\frac{1}{2}}} \\ c &= \frac{\tanh\lambda-\lambda\operatorname{sech}^2\lambda}{\lambda^2(\tanh\lambda+\lambda\operatorname{sech}^2\lambda)} \end{aligned}\right\} \quad \text{for} \quad 0<\lambda<\infty. \tag{28}$$

This is represented in figure 7, which evidently confirms our above statements.

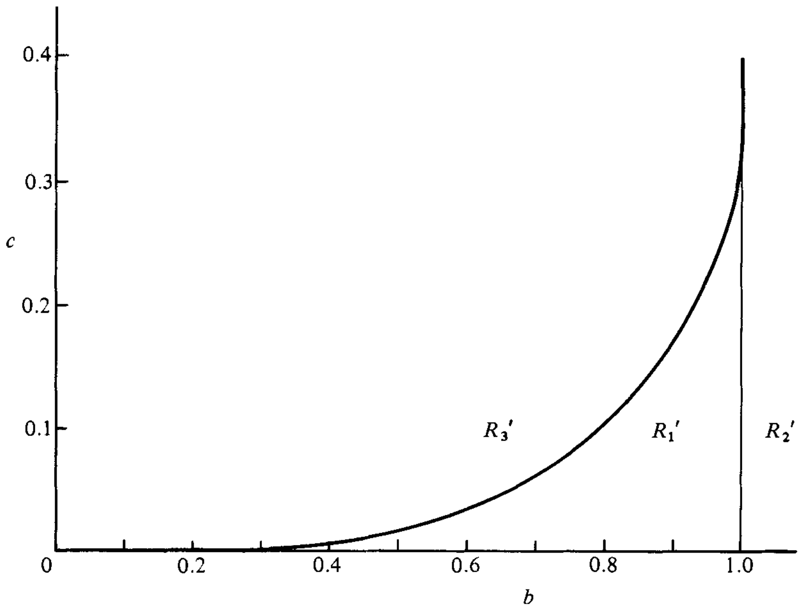

FIGURE 7. The section of the critical surface for $a = 0$.

Now we wish to consider the case $\omega = 0$. The relevant frequency equation is $\sigma = \kappa V$. It is easy to see that it always has the root zero and a maximum of two non-zero roots α and β ($\alpha < \beta$), say. The critical condition for the present case has already been represented in figure 7. Then the following statements regarding the steady-state waves can be easily verified. The α-wave, which is the original gravity wave, exists in the downstream side, while the β-wave, which is the capillary wave, exists in the upstream side. For points (b, c) in R_1' both the waves exist, for (b, c) lying in R_2' only the β-wave exists, while for points (b, c) in R_3' none of the waves propagates.

This research was supported by Grant A5294 from the Natural Sciences and Engineering Research Council of Canada. The authors are grateful for valuable comments by the referees. These comments were helpful in improving the paper.

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